Measurement of System Inertia Tensors in Precessing Nonrigid Systems

JAMES P. VANYO*
University of California, Santa Barbara, Calif.

Theme

ALTHOUGH a considerable body of literature exists for the motions of precessing nonrigid systems, there have been relatively few attempts at formulating or measuring their system inertia tensors. One motivation for doing so is to achieve equations that are more easily included in the over-all transfer function of a space vehicle than is the case for the equations of a continuously deformable medium, e.g., a liquid fuel. This paper presents an apparatus and experimental technique for measuring system inertia tensors of precessing nonrigid systems and includes preliminary results for liquids in a filled precessing spherical cavity.

Contents

The use of equivalent rigid bodies to model dynamically the motion of fluids was initiated by Green in 1833 and by Stokes in 1843. The first general study of the dynamics of a rigid body with liquid-filled cavities was performed by N. Y. Zhukovskiy[1] in 1885. He treated the case of cavities filled with a homogeneous, incompressible, inviscid fluid and proved that: "The effect of a fluid mass having zero initial velocity is identical to the effect of some equivalent rigid body; a fluid mass with an initial velocity in a multiply connected cavity also performs an action that is similar to the action of some rotor attached to the rigid body." Zhukovskiy referred to the solid body with the fluid replaced by an equivalent body as a "transformed body". Early researchers did not include the effects of viscosity or turbulence. In the case of a spherical cavity filled with an inviscid liquid, Zhukovskiy's Theorem results in a point mass centered in the cavity.

Later researchers include Munk and MacDonald,[2] who discussed the use of astronomical data to compute inertia parameters for the Earth, and Dodge,[3] who reviewed the use of mechanical models (using systems of springs, masses, and pendulums) to simulate the dynamic effect of liquids in filled and nonfilled containers. Chernous'ko[4] in 1967 examined the stability of a rigid body with an ellipsoidal cavity filled with a viscous liquid at large Reynolds numbers and computed an inertia tensor by integration over rigid and fluid elements.

Vanyo and Likins[5] presented a model for the special case of fully turbulent flow in a liquid-filled precessing spherical cavity. The model treats the interior liquid as a rigid sphere coupled to the cavity wall by an Ekman layer of the viscous liquid. It permits the liquid to be included in the cavity dynamics as an equivalent rigid body. Vanyo[6] defined the problem of a system inertia tensor experimentally and took some preliminary data for a system consisting of a rigid precessing body having a liquid-filled spherical cavity. The remainder of this paper presents these results and the experimental studies in progress.

Received August 7, 1972; synoptic received December 8, 1972. Full paper available from National Technical Information Service, Springfield, Va., 22151, as N73-14442 at the standard price (available upon request).

Index categories: Spacecraft Attitude Dynamics and Control; Structural Dynamic Analysis; Hydrodynamics.

* Assistant Professor, Department of Mechanical Engineering.

The experimental apparatus is shown schematically in Fig. 1. It consists of an outer gimbal that rotates (precesses) at a rate $\dot{\phi}$ relative to an inertially fixed axis, an inner gimbal that can be adjusted to provide a half-coning angle θ, and a spin-body assembly which rotates (spins) at the rate $\dot{\psi}$ relative to the inner gimbal. Operation is based on the principle that an inertially axisymmetric spin-body assembly, with given transverse and axial moments of inertia (I_t and I_a), will spin and precess about its mass center without vibration at parameters θ, $\dot{\phi}$, and $\dot{\psi}$ appropriate for a torque-free body. A necessary design objective of the structure was to eliminate the outer and inner gimbals from the dynamics of the spinning and precessing spin-body assembly, so that any vibration of the total apparatus could only be due to gyroscopic effects of the spin-body assembly.

To achieve this design objective, the apparatus is constructed as follows. 1) The outer gimbal is statically and dynamically balanced about the $\dot{\phi}$ axis. 2) The mass center of the inner gimbal is positioned at the intersection of the $\dot{\phi}$ and $\dot{\psi}$ axes, and the $\dot{\psi}$ and e_2 axes are principal and equal axes of inertia. Here the e_2 axis lies in the gimbal plane and is perpendicular to the $\dot{\psi}$ axis. These conditions insure that the $\dot{\phi}$ axis is always a principal axis of inertia for the inner and outer gimbals. 3) The spin-body assembly, which consists of all parts of the apparatus that spin at the rate $\dot{\psi}$, has its mass center located at the intersection of the $\dot{\phi}$ and $\dot{\psi}$ axes and is inertially axisymmetric about the $\dot{\psi}$ axis. Values of I_t and I_a for the spin-body assemblies are measured using a torsion rod and frequency of oscillation data. The first test spin-body assembly with a rigid epoxy tank containing a 22-cm-diam cavity, centrally located, had a measured ratio of $I_t/I_a = 2.846$.

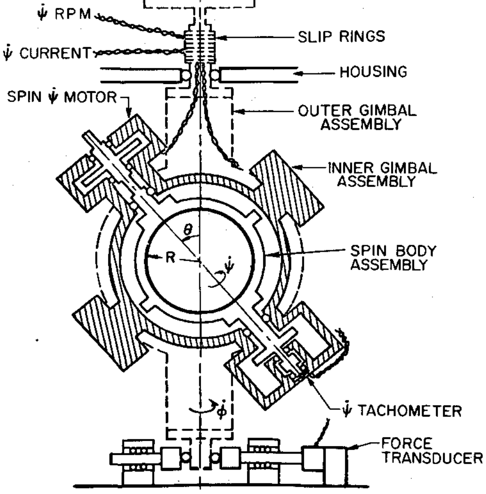

Fig. 1 Schematic drawing of the experimental apparatus.

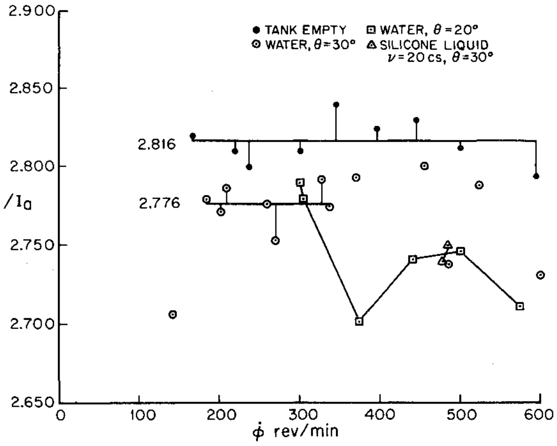

Fig. 2 Experimentally determined values of I_t/I_a for the 22-cm-diam cavity spin-body with and without liquids as noted.

Motion of a rigid body in general rotation about its mass center is defined by

$$\mathbf{M} = \dot{\mathbf{H}} \tag{1}$$

where $\mathbf{M}$ represents externally applied moment, and $\dot{\mathbf{H}}$ is the time derivative of angular momentum. Equation (1) can be expanded as components in a reference frame (e) fixed in the inner gimbal, with the e_3 axis coinciding with the $\dot\psi$ axis, the e_2 axis lying in the gimbal plane as noted earlier, and the e_1 axis perpendicular to the gimbal plane. The angles ϕ, θ, and ψ are a set of Euler angles, and M_{1e} is the component of moment along the e_1 axis, etc. Equation (1) for the spin-body assembly, with $\dot\theta = \ddot\psi = \ddot\phi = 0$, becomes

$$M_{1e}/I_a = \dot\phi \sin\theta[\dot\psi - \dot\phi(I_t/I_a - 1)\cos\theta] \tag{2}$$

with $M_{2e} = M_{3e} = 0$. If $\dot\phi = 0$, if $\theta = 0$ or π, or if

$$\dot\psi = \dot\phi(I_t/I_a - 1)\cos\theta \tag{3}$$

then M_{1e} also equals zero. This derivation verifies moment-free rotation of a spin-body assembly for values of I_t, I_a, $\dot\phi$, $\dot\psi$, and θ that satisfy Eq. (3). Included are the experimental constraints that $\dot\phi$ and $\dot\psi$ be constant and θ be fixed. In addition, Eq. (2) can be used to separate specific values of I_t and I_a from the ratio I_t/I_a.

In the series of tests reported here, the assembled apparatus was suspended from a rope. With θ set at some angle, say 30°, only those values of $\dot\phi$ and $\dot\psi$ appropriate to Eq. (3) and the spin-body ratio I_t/I_a could yield vibrationless motion. Values of $\dot\phi$ were established, and then $\dot\psi$ was varied until the entire apparatus rotated without vibration. Nine sets of values of $\dot\phi$ and $\dot\psi$ that gave vibrationless motion for the empty cavity spin-body were taken, and corresponding values of I_t/I_a were computed. The mean value was 2.816 with a one standard deviation (1σ) of 0.0137. This can be compared to the value of 2.846 obtained from frequency of oscillation data. The difference is about 1%, thus verifying the accuracy of I_t/I_a data obtained using Eq. (3). The new design of Fig. 1 forces the spin-body to precess about an inertial axis and measures values of M_{1e} using the force transducer. The torque-free mode is reached when $M_{1e} = 0$.

Several test procedures are available 1) Establish vibrationless motion ($M_{1e} = 0$) for a given nonrigid system and compute the ratio I_t/I_a. 2) Measure the contribution of a nonrigid element to the I_t/I_a of a larger rigid component. A liquid-filled spherical cavity presents this problem. The apparatus is operated first empty and then filled. The difference between the two ratios I_t/I_a is due to the liquid's participation in the system inertia tensor. 3) Determine separate values of I_t and I_a using measured values of M_{1e} at two operating conditions that deviate slightly from the vibrationless torque-free mode. A requirement here, of course, is that the system values of I_t and I_a remain essentially constant over the parameter region near the free-body mode.

Three sets of data were taken using the 22-cm-diam spherical cavity filled with liquids. Figure 2 shows the computed values of I_t/I_a for this data plotted as a function of $\dot\phi$. Also shown are the I_t/I_a values with the spin-body empty. Variations in the empty spin-body results are due to randomicity in the data (I_t/I_a does not change for the rigid system). However note that the I_t/I_a results for the water-filled system at $\theta = 30°$ also appear to have no consistent dependence on $\dot\phi$ for the seven data points connected by the horizontal line. These seven data points were also the most accurate based on the small values for estimated $+\Delta\dot\psi$ rpm that resulted in an observable vibration. There do, however, appear to be variations in I_t/I_a as functions of θ and ν. This data, because of its apparent independence of $\dot\phi$ and $\dot\psi$, permits the computation of a mean value $I_t/I_a = 2.776$ with $1\sigma = 0.0117$ and is an example of procedure 1. The difference between the results with and without the water is 0.040 ($1\sigma = 0.018$). This represents the contribution of the water to the inertia tensor of the system and is an example of procedure 2. The value of 0.040 is about 2σ and probably represents a lower limit for useful results. The new apparatus is designed to give $1\sigma \sim 0.005$.

The measured values of $I_t = 0.4181$ and $I_a = 0.1484$ in.-lb-sec^2 can be used to compute the moment of inertia (I_f) of a spherical shell of water which, if rigid and fastened to the cavity wall, would cause the spin-body assembly to vary from $I_t/I_a = 2.816$ to 2.776. The value of I_f for such a shell is 0.0034 in.-lb-sec^2. The thickness of the shell is $t = 0.0124$ in. This computation for t implies several assumptions. The critical assumption, based on the constancy of I_t/I_a over $\dot\phi$, is an inference that the fluid has an equivalent rigid mass. Zhukovskiy, in his theorem, indicated that the moment of inertia of a transformed body was independent of both time and the particular type of motion being considered, but was, of course, speaking of an inviscid fluid. It is interesting to see an apparently similar result for water in turbulent motion in a precessing spherical cavity.

It may be possible to analyze t as an angular momentum boundary-layer thickness because it is related to the participation of the liquid in the angular momentum of the spin-body assembly. The work of Rott and Lewellen[7] may be significant in such an analysis. As a check, t may also be compared to an Ekman layer thickness (h) using an equation given in Ref. 5. Over the range of ($\dot\phi$, $\dot\psi$) values averaged to compute t, the value of h ranges from 0.0062 to 0.0083 in. or from $\frac{1}{2}$ to $\frac{2}{3}$ the value of t. A planned series of tests using the improved instrumentation should provide more definitive results.

References

[1] Zhukovskiy, N. Y., *On the Motion of a Rigid Body with Cavities Filled with a Homogeneous Liquid Drop*, Collected works, Moscow, 1948; also Moiseyev, N. N. and Rumyantsev, V. V., *Dynamic Stability of Bodies Containing Fluid*, English translation, edited by H. N. Abramson, Springer-Verlag, New York, 1968, pp. x, 51, and 336.

[2] Munk, W. H. and MacDonald, G. J. F., *The Rotation of the Earth a Geophysical Discussion*, 1st ed., Cambridge University Press, London, 1960.

[3] Dodge, F. T., "Analytical Representation of Lateral Sloshing by Equivalent Mechanical Models," *The Dynamic Behavior of Liquids in Moving Containers*, edited by H. N. Abramson, NASA SP-106, 1966, Chap. 6.

[4] Chernous'ko, F. L., "Motion of a Body with a Cavity Filled with a Viscous Fluid at Large Reynolds Numbers," English translation, *Journal of Applied Mathematics and Mechanics*, Vol. 30, No. 3, March 1967, pp. 568–589.

[5] Vanyo, J. P. and Likins, P. W., "Rigid-Body Approximations to Turbulent Motion in a Liquid-Filled Precessing Spherical Cavity," *Journal of Applied Mechanics*, Vol. 39, Ser. E, No. 1, March 1972, pp. 18–24.

[6] Vanyo, J. P., "Energy Dissipation in a Liquid-Filled Precessing Spherical Cavity and Application to Satellites," Doctoral dissertation, Dec. 1969, Univ. of California, Los Angeles; University Microfilms, Ann Arbor, Mich.

[7] Rott, N. and Lewellen, W. S., "Boundary Layers and Their Interactions in Rotating Flows," *Progress in Aeronautical Sciences*, Vol. 7, 1966, pp. 111–144.